UNCLASSIFIED

AD 288 929

Reproduced
by the

ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA

UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

63-1-4

D1-82-0201

CATALOGED BY ASTIA
AS AD No. 2 88 929

"Also available from the author"

BOEING SCIENTIFIC RESEARCH LABORATORIES

A Note on Generating Chi Random Numbers

ASTIA
NOV 26 1962
TISIA A

M. D. MacLaren

Mathematics Research

September 1962

288 929

D1-82-0201

A NOTE ON GENERATING CHI RANDOM NUMBERS

by

M. D. MacLaren

Mathematical Note No. 270

Mathematics Research Laboratory

BOEING SCIENTIFIC RESEARCH LABORATORIES

September 1962

Introduction

Marsaglia [1] has given a simple method for generating exponential random numbers on a digital computer. We present a similar method for generating random numbers with the chi distribution. Such random numbers may be used to generate normal random numbers.

I. The chi distribution (of rank two) F is

$$F(a) = 0, \quad a \leq 0,$$

$$F(a) = 1 - e^{-a^2/2}, \quad 0 \leq a.$$

Let x be a random variable with the distribution F. Let $G_c(a) = \text{Prob}(x \leq a | x \leq c)$ where $0 < c$. Then for $0 \leq a \leq c$,

$$G_c(a) = (1 - e^{-a^2/2})/(1 - e^{-c^2/2})$$

$$(1) \qquad = 1 - \sum_{k=1}^{\infty} q_k (1 - a^2/c^2)^k,$$

where

$$q_k = (c^2/2)^k / [k!(e^{c^2/2} - 1)].$$

Let $\quad H_c(a) = 1 - e^{-(a^2-c^2)/2}, \quad c \leq a,$

$$= 0, \qquad a \leq c.$$

Then $\quad F(a) = (1 - e^{-c^2/2})G_c(a) + e^{-c^2/2}H_c(a).$

Thus a random number with the distribution F may be generated as follows. Generate a uniform random number u, i.e., a random number uniformly distributed on $(0,1)$. If $u < (1 - e^{-c^2/2})$,

generate a random number with the distribution G_c; otherwise generate one with distribution H_c.

A random number y with the distribution H_c may be generated by setting $y = \sqrt{ar + c^2}$, where r is a random number with the exponential distribution.

A random number x with the distribution G_c can be generated by setting

$$x = c \cdot \min[\max(u_1,u_2),\ldots,\max(u_{2z-1},u_{2z})],$$

where the u_i are independent uniform random numbers, and z is a random integer taking on the value k with probability q_k. This fact is easily verified by noting that the distribution of x is just the series (1).

For a binary computer the best choice for c is $c = 2$. On the IBM 7090 computer the average time to generate a chi random number x by this method is 112 cycles. (A cycle is 2.14 microseconds on this computer). This assumes that the exponential random numbers are generated by the method given in [1].

If x is generated by setting $x = \sqrt{2r}$, the average time is 165 cycles.

II. To generate normal random numbers we make use of the following well-known fact. Let (α,β) be the rectangular coordinates of a random point uniformly distributed on the unit circle. Then if x is a chi random number $y = \alpha x$ and $z = \beta x$ are independent standard normal random numbers.

The following methods for generating such a pair (α,β) are well known.

Method 1. Test independent pairs of uniform numbers (u,v) until a pair is found which satisfies $u^2 + v^2 \leq 1$. Then set $\alpha = u/\sqrt{u^2 + v^2}$ and $\beta = v/\sqrt{u^2 + v^2}$.

Method 2. Test independent pairs (u,v) until a pair is found which satisfies $u^2 + v^2 \leq 1$. Then set $\alpha = 2uv/(u^2 + v^2)$ and $\beta = (v^2 - u^2)/(u^2 + v^2)$.

To generate normal random numbers we can use the following procedure. Generate a chi random number x. If $x < c$, use method 2 to generate (α,β). If $c \leq x$ use method 1 to generate (α,β). Note that we can decide if $c \leq x$ before we set $x = \sqrt{c^2 + 2r}$. Therefore this square root operation can be combined with that used to generate (α,β). In effect when $c \leq x$, we compute a pair of normal random numbers y and z by

$$y = x\alpha = u[(2r + c^2)/(u^2 + v^2)]^{\frac{1}{2}}$$

and

$$z = x\beta = v[(2r + c^2)/(u^2 + v^2)]^{\frac{1}{2}}.$$

This procedure takes 156 cycles to generate one normal random number on the 7090.

REFERENCES

[1] G. Marsaglia, "Generating Exponential Random Variables," Ann. Math. Stat., vol. 32 (1961), pp. 899 - 900.